UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3506**

ASSESSMENT PATTERN : **MATH3506A**

MODULE NAME : **Mathematics in Biology I**

DATE : **13-May-08**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH3506A-001-EXAM-21

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) An insect population grows according to the discrete time model

$$N_{t+1} = rN_t \exp\left(-\frac{N_t}{K}\right),$$

where $r, K > 0$ are constants. Find all the steady state populations and analyse their stability. Sketch the cobweb map for (i) $r < 1$, (ii) $e < r < e^2$.

(b) In a continuous time population model with population density $N(t)$ at time t the *per capita* birth rate is

$$\beta(N) = \frac{rN^2}{K + N^3},$$

where $r, K > 0$ are constants, and the per capita death rate is a constant $d > 0$.

(i) Write down the differential equation governing the population dynamics and show that for $K < \frac{1}{2}\left(\frac{2r}{3d}\right)^3$ there are two distinct positive steady states. (Hint: First sketch the curve $F(N) = N^3 - (r/d)N^2$).

(ii) Determine the stability of these steady states.

2. In a predator-prey model, the predator P and prey N interact according to

$$\frac{dN}{dt} = N(a - bP - eN)$$
$$\frac{dP}{dt} = P(cN - fP - d)$$

where $a, b, c, d, e, f > 0$ are all constants.

(a) Briefly explain the biological interpretation of the equations.

(b) Find all the steady states, showing that a non-zero steady state (N^*, P^*) is possible only if $ac > ed$.

(c) Classify the stability of the non-zero steady state when $ac > ed$ and sketch the phase plane in this case.

(d) What happens if $e = 0$ (and $a, b, c, d, f > 0$)? Sketch the phase plane also in this case.

3. (a) Find the explicit solution of the Logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0,$$

where $r, K > 0$ are constants.

(b) By introducing the new rescaled time variable τ via $d\tau = \rho(t)dt$, or otherwise, show that if $\rho(t)$ is periodic of period T then the solution to the time-dependent logistic equation

$$\frac{dN}{dt} = \rho(t)N\left(1 - \frac{N}{K}\right)$$

is for $t = nT + s$ with $s \in [0, T)$ and n integer,

$$N(nT+s) = \frac{N_0}{\frac{N_0}{K} + e^{-nRT}\left(1 - \frac{N_0}{K}\right)\exp\left(-\int_0^s \rho(t)\,dt\right)} \tag{1}$$

where $R = \dfrac{1}{T}\displaystyle\int_0^T \rho(t)dt$.

(c) Find the asymptotic behaviour $N_\infty(s) = \lim_{n\to\infty} N(nT+s)$ in equation (1) when $\rho(t) = r\sin(\omega t)$ for $r, \omega > 0$ constants.

(d) Find also $\max_s N_\infty(s)$ for the given $\rho(t) = r\sin(\omega t)$.

4. In a simple model for a population of reptiles, the population is divided into 2 classes: the juvenile class J and the adult class A. Juveniles cannot reproduce, and all adults reproduce at the rate b_A. The probability of a newborn surviving to a juvenile is p_J, the probability of a juvenile surviving to adulthood is p_A, and the probability of an adult surviving thereafter from one time unit to the next is p_S where $p_S < p_A - p_J b_A$.

(a) Derive equations for the size of the juvenile population J_{k+1} and adult population A_{k+1} at generation $k+1$ in terms of generation $k \geqslant 0$ and find the Leslie matrix for the model.

(b) If X_k is the juvenile fraction of the population at generation k (excluding newborns), show that

$$X_{k+1} = \frac{1 - X_k}{\alpha + \beta X_k}.$$

where α, β are constants which you should find in terms of p_A, p_J, p_S and b_A.

(c) Find the long term stable population distribution for the reptiles.

5. A non-fatal virus spreads within a closed community via infected individuals who infect susceptible individuals. All infected individuals recover and then temporarily acquire immunity from the disease. Immunity only lasts for a limited period after which individuals become susceptible again. Birth and deaths are to be neglected on the model timescale. A simple model for such a process is the set of differential equations:

$$\begin{aligned}\frac{dS}{dt} &= -rSI + \gamma R \\ \frac{dI}{dt} &= rSI - aI \\ \frac{dR}{dt} &= aI - \gamma R\end{aligned}$$

(a) Give an interpretation of the model, stating what the variables S, I, R and parameters $r > 0$, $a > 0$ and $\gamma > 0$ represent.

(b) Show that the total population N over all the classes remains constant and hence obtain a coupled pair of ordinary differential equations for S, I only.

(c) Find all the steady states of the system for S, I found in part (b) and analyse their linear stability.

(d) Plot the phase plane for the system for S, I found in part (b) in the cases (i) $N < a/r$ and (ii) $N > a/r$ and comment on the results.